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Nilpotent pairs in semisimple Lie algebras and their characteristics

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Introduction

In a recent article [Gi99], V. Ginzburg introduced and studied in depth the notion of a *principal nilpotent pair* in a semisimple Lie algebra $\mathfrak{g}$. He also obtained several results for more general pairs. As a next step, we considered in [Pa99] *almost principal nilpotent pairs*. The aim of this paper is to make a contribution to the general theory of nilpotent pairs. Roughly speaking, a nilpotent pair $\mathbf{e} = (e_1, e_2)$ consists of two commuting elements in $\mathfrak{g}$ that can independently be contracted to the origin (see precise definition in sect. 1). A principal nilpotent pair is a double counterpart of a regular (=principal) nilpotent element. Consequently, the theory of nilpotent pairs should stand out as double counterpart of the theory of nilpotent orbits. As the cornerstone of the latter is the Morozov–Jacobson theorem and the concept of a characteristic, the primary goal is to realize to which extent these can be generalized to the double setting.

The fundamental distinction of the double situation is that there is no general analogue of the Morozov–Jacobson theorem. What remains true is that any nilpotent pair has a characteristic $\mathbf{h} = (h_1, h_2)$, which is unique within to conjugacy **(1.4)**. Hence characteristics can be used to study further properties of nilpotent pairs. Generalizing Dynkin's approach in [Dy52a], we prove that the number of G-orbits of characteristics of nilpotent pairs is finite **(1.6)** and provide some information about the numerical labels $\alpha_j(h_i)$, where $\{\alpha_1, \dots, \alpha_n\}$ is a suitably chosen set of simple roots of $\mathfrak{g}$ **(1.7)**. Since the number of G-orbits of nilpotent pairs is infinite (see [Gi99, 5.5]), one encounters a challenging problem to restore somehow a "one-to-one" correspondence between nilpotent pairs and characteristics. Our solution is that we introduce in sect. 2 *wonderful* (nilpotent) pairs. The definition is given in terms of the bi-grading of $\mathfrak{g}$ determined by $\mathbf{h}$ and involves only $\mathbf{h}$-eigenspaces with integral eigenvalues **(2.4)**. We prove that if $\mathbf{e}, \mathbf{e}'$ are two wonderful pairs with the same characteristic $\mathbf{h}$, then $\mathbf{e}$ and $\mathbf{e}'$ are $Z_G(\mathbf{h})$-conjugate **(2.9)**. This implies that there are finitely many G-orbits of wonderful pairs. On the other hand, specific classes of nilpotent pairs considered in [Gi99],[EP99],[Pa99] are wonderful.

In section 3, properties of several classes of wonderful pairs are studied. For instance, we call a nilpotent pair $\mathbf{e}$ *even*, if $\dim \mathfrak{z}_{\mathfrak{g}}(\mathbf{h}) = \dim \mathfrak{z}_{\mathfrak{g}}(\mathbf{e})$. It is a natural analogue of an even nilpotent element. We show that if $\mathbf{e}$ is even, then it is wonderful and the eigenvalues of $\mathbf{h}$ in $\mathfrak{g}$ are integral **(3.3)**.

In section 4, we describe characteristics of principal and almost principal nilpotent pairs **(4.2)** and indicate the relationship between eigenvalues of $\mathbf{h}$ in $\mathfrak{z}_{\mathfrak{g}}(\mathbf{e})$ and the exponents

of the corresponding Levi subalgebras $\mathfrak{z}_{\mathfrak{g}}(h_i)$ **(4.3)**.

What is still lacking is a true understanding of possible fractional eigenvalues of $\mathbf{h}$ for arbitrary wonderful pairs. Unlike the ordinary theory (= theory of nilpotent orbits), these eigenvalues may have very large denominators **(3.7)**.

The ground field $\Bbbk$ is algebraically closed and of characteristic zero. Algebraic groups are denoted by capital Roman letters, while their Lie algebras by the corresponding small Gothic letters. Throughout, $\mathfrak{g}$ is a semisimple Lie algebra and G is its adjoint group. For any set $M\subset\mathfrak{g}$, we let $\mathfrak{z}_{\mathfrak{g}}(M)$ (resp. $Z_G(M)$) denote the centralizer of M in $\mathfrak{g}$ (resp. in G) and $M^\perp$ the orthogonal complement in $\mathfrak{g}$ with respect to the Killing form. For $M=\{a,\dots,z\}$, we simply write $\mathfrak{z}_g(a,\dots,z)$ or $Z_G(a,\dots,z)$. If $N\subset G$, then $Z_G(N)$ stands for the centralizer of N in G. For $x\in\mathfrak{g}$ and $s\in G$, we write $s{\cdot}x$ in place of $(\mathrm{Ad}\, s)x$. Write A^o for the identity component of an algebraic group A.

$\langle a,\dots,z\rangle$ is the linear span of elements of a vector space;

$\mathbb{P}=\{0,1,2,\dots\}$, $\mathbb{N}=\{1,2,\dots\}$.

Our general reference for algebraic groups is Vinberg–Onishchik's book [VO88]. For nilpotent orbits consult [Conj], [VGO90, ch. 6], and [CM93].

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1 Characteristics and their properties

Let us begin with a definition, which is due to V. Ginzburg.

1.1 Definition. A pair $\mathbf{e}=(e_1,e_2)\in\mathfrak{g}\times\mathfrak{g}$ is said to be *nilpotent* in $\mathfrak{g}$, if (i) $[e_1,e_2]=0$ and (ii) for any $(t_1,t_2)\in\Bbbk^*\times\Bbbk^*$, there exists $g=g(t_1,t_2)\in G$ such that $(\,t_1e_1,\ t_2e_2\,)=(g{\cdot}e_1,\ g{\cdot}e_2\,)$.

Obviously, then both e_1 and e_2 are nilpotent elements of $\mathfrak{g}$. Note however that a nilpotent pair is not the same as a commuting pair of nilpotent elements. A nilpotent pair is said to be *trivial*, if one of e_i's is equal to 0. It was shown in [Gi99] (see also 1.4(1) below) that condition (ii) is equivalent to the following one: there exists a pair of semisimple elements $\mathbf{h}=(h_1,h_2)\in\mathfrak{g}\times\mathfrak{g}$ such that $\mathrm{ad}\,h_1$ and $\mathrm{ad}\,h_2$ have rational eigenvalues and

$$[h_1,h_2]=0,\quad [h_i,e_j]=\delta_{ij}e_j\quad (i,j\in\{1,2\})\ . \tag{1.2}$$

Ginzburg called such a pair an associated semisimple pair. He also proved that an associated semisimple is unique up to conjugacy, if $\mathbf{e}$ is a "pre-distinguished" nilpotent pair [Gi99, sect. 5]. We shall prove that, after a slight modification of the definition, the result actually holds for all nilpotent pairs. For this reason, we prefer to use the classical term 'characteristic' introduced by E. B. Dynkin in [Dy52a].

1.3 Definition. A pair of semisimple elements $\mathbf{h} \in \mathfrak{g} \times \mathfrak{g}$ is called a *characteristic* of $\mathbf{e}$, if it satisfies commutation relations (1.2) and $\{h_1, h_2\} \subset \mathfrak{z}_{\mathfrak{g}}(\mathbf{e})^{\perp}$.

Recall that if A is a linear algebraic group, then there exists an algebraic Levi decomposition $A = A^{red} \ltimes A^{nil}$, where A^{nil} is the unipotent radical and A^{red} is a reductive Levi subgroup of A (see [VO88, ch. 6]). On the Lie algebra level, this yields the semi-direct sum $\mathfrak{a} = \mathfrak{a}^{red} \oplus \mathfrak{a}^{nil}$. The following assertion is a transfer to the double setting of some properties of $\mathfrak{sl}_2$-triples. Its proof is a combination of arguments used in [Vi79, §2] and [Gi99, sect. 1].

1.4 Theorem. 1. *Each nilpotent pair has a characteristic;*
2. *If $\mathbf{h}$ and $\mathbf{h}'$ are two characteristic, then there exist $u \in Z_G(\mathbf{e})^{nil}$ such that $u{\cdot}\mathbf{h} = \mathbf{h}'$;*
3. *If $\mathbf{h}$ is any characteristic, then the eigenvalues of $\operatorname{ad} h_1$ and $\operatorname{ad} h_2$ are rational.*

Proof. 1. Consider the algebraic group $N := \{g \in G \mid g{\cdot}e_i \in \Bbbk e_i, i = 1, 2\}$. In view of condition 1.1(ii), we have the exact sequence

$$\{1\} \to Z_G(\mathbf{e}) \to N \xrightarrow{\tau} (\Bbbk^*)^2 \to \{1\} .$$

(If $g{\cdot}e_i = t_i e_i$, then $\tau(g) = (t_1, t_2)$.) Since $(\Bbbk^*)^2$ is reductive and Abelian, both N^{nil} and $[N, N]$ lie in $\operatorname{Ker} \tau$. Hence τ induces a surjective homomorphism $(N^{red}/[N^{red}, N^{red}])^o \to (\Bbbk^*)^2$. By a standard property of diagonalizable groups, this means that there exists a 2-dimensional torus $C \subset N^o$ such that $|Z_G(\mathbf{e}) \cap C| < \infty$ and $\tau|_C$ is onto. On the Lie algebra level, this yields $\mathfrak{n}^{nil} = \mathfrak{z}_{\mathfrak{g}}(\mathbf{e})^{nil}$ and $\mathfrak{n}^{red} = \mathfrak{z}_{\mathfrak{g}}(\mathbf{e})^{red} \oplus \mathfrak{c}$. The restriction of the Killing form to either of the reductive subalgebras $\mathfrak{n}^{red}$ and $\mathfrak{z}_{\mathfrak{g}}(\mathbf{e})^{red}$ is non-degenerate [VO88, ch. 4 §1.1]. Hence C can be chosen so that $\mathfrak{c} \perp \mathfrak{z}_{\mathfrak{g}}(\mathbf{e})^{red}$. As $\mathfrak{n}^{nil} \perp \mathfrak{n}$, we obtain $\mathfrak{c} \perp \mathfrak{z}_{\mathfrak{g}}(\mathbf{e})$. Restricting the differential of τ to $\mathfrak{c}$ yields an isomorphism $d\tau : \mathfrak{c} \xrightarrow{\sim} \Bbbk^2$. Define $h_1, h_2 \in \mathfrak{c}$ by $d\tau(h_1) = (1, 0)$, $d\tau(h_2) = (0, 1)$. Then h_1, h_2 have rational eigenvalues and satisfy (1.2).

2. Suppose $\mathbf{h}' = (h_1', h_2')$ is another characteristic of $\mathbf{e}$. Since $h_i' - h_i \in \mathfrak{z}_{\mathfrak{g}}(\mathbf{e})$ and $h_i' - h_i \perp \mathfrak{z}_{\mathfrak{g}}(\mathbf{e})$, we obtain $h_i' - h_i \in \mathfrak{z}_{\mathfrak{g}}(\mathbf{e})^{nil}$ $(i = 1, 2)$. Thus $\mathfrak{c}$ and $\mathfrak{c}' := \langle h_1', h_2' \rangle$ are two maximal diagonalizable subalgebras of the solvable algebra $\mathfrak{c} \oplus \mathfrak{z}_{\mathfrak{g}}(\mathbf{e})^{nil}$. By the standard conjugacy theorem (see [VO88, ch.3 §2]), there exists $u \in Z_G(\mathbf{e})^{nil}$ such that $u{\cdot}\mathfrak{c} = \mathfrak{c}'$. It then follows from (1.2) that $u{\cdot}h_i = h_i'$ $(i = 1, 2)$.

3. Because the characteristic constructed in the first part of the proof had rational eigenvalues, we conclude by the second part. □

1.5 Corollary. *Suppose $\mathbf{e}$ is a nilpotent pair and $\mathbf{h}$ is a semisimple pair satisfying (1.2). If $\mathfrak{z}_{\mathfrak{g}}(\mathbf{h}) \cap \mathfrak{z}_{\mathfrak{g}}(\mathbf{e}) = 0$, then $\mathbf{h}$ is a characteristic of $\mathbf{e}$ and $\mathfrak{z}_{\mathfrak{g}}(\mathbf{e})$ contains no semisimple elements.*

Proof. Let $\mathfrak{g} = \mathfrak{z}_{\mathfrak{g}}(\mathbf{h}) \oplus \mathfrak{m}$ be the $Z_G(\mathbf{h})$-stable decomposition. Then $\mathfrak{z}_{\mathfrak{g}}(\mathbf{e}) \subset \mathfrak{m}$ and therefore $\mathfrak{z}_{\mathfrak{g}}(\mathbf{e})$ is orthogonal to h_1, h_2. That is, $\mathbf{h}$ is a characteristic of $\mathbf{e}$. From the

proof of (1.4), it follows that $\mathfrak{z}_{\mathfrak{g}}(\mathbf{h})$ contains a reductive Levi subalgebra of $\mathfrak{z}_{\mathfrak{g}}(\mathbf{e})$. Thus, $\mathfrak{z}_{\mathfrak{g}}(\mathbf{e})^{red} = 0$. □

This theorem shows that nilpotent pairs are "sufficiently good" double analogues of nilpotent elements. Willing to extend the classical theory of nilpotent orbits and $\mathfrak{sl}_2$-triples to the double setting, it is worth to recall Dynkin's approach in 1952. Any 3-dimensional simple algebra $\mathfrak{a}$ has the unique basis $\{e, \tilde{h}, f\}$ such that $[\tilde{h}, e] = 2e, [\tilde{h}, f] = -2f, [e, f] = \tilde{h}$. The semisimple element $\tilde{h}$ is called the defining vector or characteristic of $\mathfrak{a}$. Considering G-orbits (conjugacy classes) of 3-dimensional simple subalgebras in $\mathfrak{g}$, Dynkin first proved that two such subalgebras are G-conjugate if and only if their characteristics are [Dy52a, Th. 8.1]. Second, he proved that if $\tilde{h}_+$ is the dominant representative in $G{\cdot}\tilde{h}$ (relative to a fixed set of simple roots $\alpha_1, \dots, \alpha_n$), then $\alpha_i(\tilde{h}_+) \in \{0, 1, 2\}$ [Dy52a, Th. 8.3]. This readily yields finiteness for the number of G-orbits of characteristics and hence of nilpotent orbits. According to the modern terminology, $\{e, \tilde{h}, f\}$ is called an $\mathfrak{sl}_2$-triple and $\tilde{h}$ is also called a characteristic of e.
It turns out that the second part of the above program has a counterpart in the double setting. Let us fix some relevant notation and terminology. Suppose $\mathbf{h}$ is a characteristic of $\mathbf{e}$. We shall consider the bi-grading of $\mathfrak{g}$ determined by $\mathbf{h}$: $\mathfrak{g} = \bigoplus_{p,q} \mathfrak{g}_{p,q}$, where $\mathfrak{g}_{p,q} = \{x \in \mathfrak{g} \mid [h_1, x] = px, [h_2, x] = qx\}$ and (p, q) ranges over a finite subset of $\mathbb{Q} \times \mathbb{Q}$ containing $(0, 0)$, $(1, 0)$, $(0, 1)$. The pairs (p, q) with $\mathfrak{g}_{p,q} \neq 0$ will be referred to as the *eigenvalues* of $\mathbf{h}$ in $\mathfrak{g}$. An eigenvalue (p, q) is said to be *integral*, if $p, q \in \mathbb{Z}$. Otherwise it is called *fractional*. The same terminology is used for the corresponding eigenspaces.

1.6 Theorem. *There exist finitely many G-orbits of characteristics of nilpotent pairs.*

Proof. 1. Arguing by induction on $\dim \mathfrak{g} + \mathrm{rk}\, \mathfrak{g}$, we first note that the claim is true for $\mathfrak{sl}_2$.

2. Assuming that $\mathbf{h}$ has fractional eigenvalues, one may replace $\mathfrak{g}$ by the smaller semisimple subalgebra $[\tilde{\mathfrak{g}}, \tilde{\mathfrak{g}}]$, where $\tilde{\mathfrak{g}} = \oplus_{p,q \in \mathbb{Z}} \mathfrak{g}_{p,q}$. Indeed, $\tilde{\mathfrak{g}}$ is reductive and $e_1, e_2, h_1, h_2 \in \tilde{\mathfrak{g}}$. Then, being nilpotent, e_1, e_2 belong to $[\tilde{\mathfrak{g}}, \tilde{\mathfrak{g}}]$. Since h_1, h_2 are orthogonal to $\mathfrak{z}_{\mathfrak{g}}(\mathbf{e}) \cap \tilde{\mathfrak{g}}$ and the latter contains the centre of $\tilde{\mathfrak{g}}$ (if any), we have $h_1, h_2 \in [\tilde{\mathfrak{g}}, \tilde{\mathfrak{g}}]$. Thus, $\mathbf{h}$ is a characteristic of $\mathbf{e}$ relative to $[\tilde{\mathfrak{g}}, \tilde{\mathfrak{g}}]$ and, by the inductive assumption, $[\tilde{\mathfrak{g}}, \tilde{\mathfrak{g}}]$ contains finitely many $[\tilde{G}, \tilde{G}]$-orbits of characteristics. Clearly these orbits generate finitely many G-orbits in $\mathfrak{g} \times \mathfrak{g}$.

3. Assume now that $\mathfrak{g} = \tilde{\mathfrak{g}}$, i.e., the eigenvalues of $\mathbf{h}$ are integral. Fix a Cartan subalgebra $\mathfrak{t}$ of $\mathfrak{g}$ such that $\{h_1, h_2\} \subset \mathfrak{t}$. Then $\mathfrak{t} \subset \mathfrak{g}_{0,0}$. Choose a set of simple roots Π with respect to $\mathfrak{t}$ so that $h_2 + \varepsilon h_1$ is dominant for all sufficiently small positive $\varepsilon \in \mathbb{Q}$. For all $\alpha \in \Pi$, we then have $\alpha(h_2) \geq 0$ and if $\alpha(h_2) = 0$, then $\alpha(h_1) \geq 0$.

4. Assume that $\beta(h_2) \geq 2$ for some $\beta \in \Pi$. Then we may just throw it away! That is, consider $\Pi' = \Pi \setminus \{\beta\}$ and the corresponding Levi subalgebra $\mathfrak{g}' \subset \mathfrak{g}$. The constraint on β implies that $\mathfrak{g}'_{p,q} = \mathfrak{g}_{p,q}$ for $(p, q) \in \{(0, 0), (1, 0), (0, 1)\}$. Hence $e_1, e_2, h_1, h_2 \in \mathfrak{g}'$

and, as in part 2, we even have $e_1, e_2, h_1, h_2 \in [\mathfrak{g}', \mathfrak{g}']$. Thus, we may apply the inductive assumption to $[\mathfrak{g}', \mathfrak{g}']$.

5. Certainly, the constraint that $\alpha(h_2) \in \{0, 1\}$ for all $\alpha \in \Pi$ leaves no much room and one gets only finitely many possibilities for h_2. By symmetry, the same argument applies to h_1. It follows that, up to conjugacy, there are finitely many possibilities for $\mathbf{h}$. This completes the proof. $\square$

Instead of appealing to the symmetry, one can exploit another argument in the last part of the proof, keeping the same choice of Π. This has an advantage of giving a more precise information about numerical labels $\alpha(h_i)$ $(i = 1, 2)$. To this end, recall that a subalgebra $\mathfrak{s} \subset \mathfrak{g}$ is called *regular*, if its normalizer $\mathfrak{n}_{\mathfrak{g}}(\mathfrak{s})$ contains a Cartan subalgebra. Set $\mathfrak{l}_i = \mathfrak{z}_{\mathfrak{g}}(h_i)$, $i = 1, 2$.

1.7 Theorem. *Let* $\mathbf{e}$ *be a nilpotent pair with a characteristic* $\mathbf{h}$. *Suppose* $\{e_1, e_2\}$ *is not contained in a proper regular semisimple subalgebra of* $\mathfrak{g}$. *Fix a Cartan subalgebra* $\mathfrak{t}$ *containing* $\{h_1, h_2\}$. *Then there exists a set of simple roots* Π *relative to* $\mathfrak{t}$ *such that*

(i) $\alpha(h_2) \in \{0, 1\}$ *for all* $\alpha \in \Pi$*;*

(ii) *If* $\alpha(h_2) = 0$, *then* $\alpha(h_1) \in \{0, 1\}$*;*

(iii) *If* $\alpha(h_2) = 1$, *then* $\alpha(h_1) \in \{d, \dots, -1, 0\}$, *where* d *is a (negative) constant depending only on* $\mathfrak{l}_2$.

Proof. 1. Inductive steps used in the proof of Theorem 1.6 provide us with regular subalgebras in $\mathfrak{g}$. Hence, under our hypothesis on $\mathbf{e}$ and with the same choice of $\mathfrak{t}$ and Π, we already see that the eigenvalues of $\mathbf{h}$ must be integral, $\alpha(h_2) \in \{0, 1\}$, and $\alpha(h_1) \geq 0$, if $\alpha(h_2) = 0$.
2. If either $\alpha(h_1) \geq 2$ or $\alpha(h_1) = \alpha(h_2) = 1$ for some $\alpha \in \Pi$, one can again, as in the proof of (1.6), throw away this α and get a regular semisimple subalgebra containing e_1, e_2. Thus, this cannot occur.
3. It remains to obtain the lower bound for $\alpha(h_1)$, if $\alpha(h_2) = 1$. Notice that h_1 induces in $\mathfrak{l}_2$ the $\mathbb{Z}$-grading $\mathfrak{l}_2 = \oplus_{p \in \mathbb{Z}} \mathfrak{g}_{p,0}$ and that $\Pi_2 := \{\alpha \in \Pi \mid \alpha(h_2) = 0\}$ is a set of simple roots for $\mathfrak{l}_2$. We also know that $\alpha(h_1) \leq 1$ for $\alpha \in \Pi_2$. Look what is happening on the next level. Suppose $\mu \in \Pi$ is such that $\mu(h_2) = 1$, and let e_μ be a nonzero root vector. Then e_μ is a *lowest* weight vector of an irreducible $\mathfrak{l}_2$-module. Denote this module by $\mathbb{V}_\mu$. We have $\mathbb{V}_\mu \subset \oplus_{p \in \mathbb{Z}} \mathfrak{g}_{p,1}$. Let λ be the root of $\mathfrak{g}$ corresponding to the highest weight of $\mathbb{V}_\mu$. Then, more precisely,

$$\mathbb{V}_\mu \subset \bigoplus_{p=\mu(h_1)}^{\lambda(h_1)} \mathfrak{g}_{p,1} .$$

Let us say that $\mathrm{T}(\mathbb{V}_\mu) := \lambda(h_1) - \mu(h_1)$ is the height of $\mathbb{Z}$-grading on $\mathbb{V}_\mu$. If $\lambda(h_1) < 0$, then $e_2 \notin \mathbb{V}_\mu$ and, as above, we could drop the simple root μ. Under our assumption, this is however impossible and we must have $\lambda(h_1) \geq 0$. It is thus enough to give an upper bound on $\lambda(h_1) - \mu(h_1)$ for any such μ. The problem can be stated as follows:

The algebra $\mathfrak{l}_2$ and an $\mathfrak{l}_2$-module $\mathbb{V}_\mu$ are compatibly $\mathbb{Z}$-graded; the grade of each simple root in $\mathfrak{l}_2$ is either 0 or 1. Give an upper bound on the height of the $\mathbb{Z}$-grading on $\mathbb{V}_\mu$.

The answer was essentially given by Dynkin, who considered the height of representation in case, where all simple roots are of grade 1 [Dy52b, Suppl., §2 n^o 12]. (Actually, Dynkin considered not gradings but the associated partitions of the weights of a representation into 'layers', the height being {the number of layers} −1.) If some of the simple roots have grade 0, then the height can only decrease. Hence the bound given by Dynkin applies in our situation as well. Being adapted to our setting, it reads $\mathrm{T}(\mathbb{V}_\mu) \le -2(\mu|\nu)$, where ν is the sum of all positive coroots of $\mathfrak{l}_2$ and $(\cdot|\cdot)$ is a Weyl group invariant inner product on $\mathfrak{t}^*$. Whence $\mu(h_1) \ge 2(\mu|\nu)$, and we may take $d = \min_{\mu\in\Pi\setminus\Pi_2} 2(\mu|\nu)$. □

Remark. In section 4, we describe some classes of nilpotent pairs that do not lie in proper regular semisimple subalgebras and give a better expression for d in those cases.

Unfortunately, attempts to extend further the Morozov-Jacobson theorem to arbitrary nilpotent pairs fail: It is impossible in general to introduce an "opposite" nilpotent pair $\mathbf{f}$. And what is worse, the number of G-orbits of all nilpotent pairs is infinite (see [Gi99, 5.5]). Comparing with Theorem 1.6 implies that there should exist infinite families of G-orbits of nilpotent pairs with the same characteristic. Actually, it is not hard to realize that nilpotent pairs described in [loc. cit] give examples of such families. Therefore one can address the following somewhat vague problem:

(1.8) *Find a natural subclass of all nilpotent pairs that consists of finitely many orbits, admits a rich structure theory, and includes all interesting examples.*

Let us describe available 'interesting' examples. We first indicate the most restrictive case, where the opposite $\mathbf{f}$ exists and the theory becomes entirely parallel to the ordinary one. By the invariance of the Killing form, $\mathfrak{z}_{\mathfrak{g}}(\mathbf{e})^\perp = [\mathfrak{g}, e_1] + [\mathfrak{g}, e_2]$. That is, $h_i \in \mathrm{Im}\,(\mathrm{ad}\, e_1) + \mathrm{Im}\,(\mathrm{ad}\, e_2)$, if $\mathbf{h}$ is a characteristic of $\mathbf{e}$.

1.9 Proposition. *The following conditions are equivalent:*

1. $h_1 \in \mathrm{Im}\,(\mathrm{ad}\, e_1)$;
2. $h_2 \in \mathrm{Im}\,(\mathrm{ad}\, e_2)$;
3. *there exist commuting $\mathfrak{sl}_2$-triples $\{e_1, 2h_1, f_1\}$ and $\{e_2, 2h_2, f_2\}$.*

Proof. Since (3) implies (1) and (2), it suffices to demonstrate that (1)⇒(3).
Suppose $h_1 = [e_1, f]$. Then replacing f by its projection to the (-1)-eigenspace of $\mathrm{ad}\, h_1$, we obtain the $\mathfrak{sl}_2$-triple $\{e_1, 2h_1, f_1\}$. Further, $0 = [e_2, h_1] = [e_1, [e_2, f_1]]$. Hence $[e_2, f_1]$ lies in the (-1)-eigenspace of $\mathrm{ad}\, h_1$ in $\mathfrak{z}_{\mathfrak{g}}(e_1)$. As the latter is positively graded, this forces $[e_2, f_1] = 0$. Similarly, $[h_2, f_1] = 0$. Thus e_2, h_2 belong to the reductive subalgebra $\mathfrak{k}_1 := \mathfrak{z}_{\mathfrak{g}}(e_1, h_1, f_1)$. Since the restriction of the Killing form to $\mathfrak{k}_1$ is non-degenerate, it follows from the condition $h_2 \perp \mathfrak{z}_{\mathfrak{g}}(\mathbf{e})$ that h_2 is orthogonal to $\mathfrak{z}_{\mathfrak{k}_1}(e_2)$ and hence $h_2 \in [\mathfrak{k}_1, e_2]$. Thus, e_2 and $2h_2$ can be included into the $\mathfrak{sl}_2$-triple inside of $\mathfrak{k}_1$. □

Remark. Each member of a nilpotent pair has a characteristic in its own right. To

avoid a confusion between characteristics of nilpotent pairs and of nilpotent elements, the latter are always marked with 'tilde'. We shall say that $\mathbf{e}$ is *rectangular* whenever the equivalent conditions of (1.9) hold. Roughly speaking, the proposition claims that $\mathbf{e}$ is rectangular if and only if $2h_i = \tilde{h}_i \ (i = 1, 2)$.

The number of G-orbits of rectangular pairs is evidently finite, and various assertions for such pairs can immediately be derived on the base of the ordinary theory (see [EP99], [Pa99, sect. 3]). Yet, there exist a plenty of interesting not necessarily rectangular pairs: *principal* [Gi99] and *almost principal* [Pa99] nilpotent pairs. A nilpotent pair $\mathbf{e}$ is called principal (resp. almost principal) if $\dim \mathfrak{z}_{\mathfrak{g}}(\mathbf{e}) = \mathrm{rk}\, \mathfrak{g}$ (resp. $\mathrm{rk}\, \mathfrak{g} + 1$). Dealing with these pairs, we shall usually omit the adjective 'nilpotent'. Finiteness of the number of G-orbits of principal pairs was proved in [Gi99, 3.9]. Below, we show this holds for the almost principal pairs. A variety of other results obtained for principal and almost principal pairs allows us to treat them as 'very interesting'. It is therefore natural to require that a subclass we are searching for would include all rectangular, principal, and almost principal pairs.

2 Wonderful nilpotent pairs

In this section we present our solution to problem 1.8. The idea to consider filtrations and limits associated with nilpotent elements is due to R. Brylinski [Br89]. Then V. Ginzburg realized that this nicely works in case of nilpotent pairs.

Let $\mathbf{e}$ be a nilpotent pair in $\mathfrak{g}$ and $\mathbf{h}$ a characteristic of it. Put $\mathbb{P} := \{0, 1, 2, \ldots\}$. Making use of $\mathbf{e}$, one may define three filtrations for any subspace $M \subset \mathfrak{g}$:

- e_1-filtration: $M(i, *) = \{x \in M \mid (\mathrm{ad}\, e_1)^{i+1} x = 0\},\ i \in \mathbb{P}$;
- e_2-filtration: $M(*, j) = \{x \in M \mid (\mathrm{ad}\, e_2)^{j+1} x = 0\},\ j \in \mathbb{P}$;
- $\mathbf{e}$-filtration: $M(i, j) = M(i, *) \cap M(*, j)$.

The corresponding limits are defined by the formulas:

$\lim_{e_1} M = \sum_{i\in\mathbb{P}} (\mathrm{ad}\, e_1)^i M(i, *) \subset \mathfrak{g}$,
$\lim_{e_2} M = \sum_{j\in\mathbb{P}} (\mathrm{ad}\, e_2)^j M(*, j) \subset \mathfrak{g}$,
$\lim_{\mathbf{e}} M = \sum_{i,j\in\mathbb{P}} (\mathrm{ad}\, e_1)^i (\mathrm{ad}\, e_2)^j M(i, j) \subset \mathfrak{g}$.

Notice that the order of $\mathrm{ad}\, e_1$ and $\mathrm{ad}\, e_2$ in the last line is immaterial, since $[e_1, e_2] = 0$. It follows that $\lim_{e_i} M \subset \mathfrak{z}_{\mathfrak{g}}(e_i)$ and $\lim_{\mathbf{e}} M \subset \mathfrak{z}_{\mathfrak{g}}(\mathbf{e})$. It is easily seen that in all three cases $\dim(\lim_{\bullet} M) \le \dim M$ and the equality is equivalent to the fact that the sum in the definition of corresponding limit is actually direct. (To check this for the $\mathbf{e}$-limit, one should use the equality $\dim (\mathrm{ad}\, e_1)^i (\mathrm{ad}\, e_2)^j M(i, j) =$
$\dim M(i, j) - \dim M(i-1, j) - \dim M(i, j-1) + \dim M(i-1, j-1)$.)
We shall repeatedly use the following sufficient condition for this to happen. In case of ordinary filtrations this was observed in [Br89]. Recall that $\mathfrak{l}_i = \mathfrak{z}_{\mathfrak{g}}(h_i)$.

2.1 Lemma.
1. *If* $M \subset \mathfrak{l}_i$, *then* $\dim(\lim_{e_i} M) = \dim M$ *(*$i = 1, 2$*);*
2. *If* $M \subset \mathfrak{z}_{\mathfrak{g}}(\mathbf{h})$, *then* $\dim(\lim_{\mathbf{e}} M) = \dim M$.

Proof. Relations 1.2 show that different summands in definition of all limits belong to different eigenspaces (relative to h_i and $\mathbf{h}$ respectively). □

If $A = \oplus_i A_i$ is a $\mathbb{Q}$-graded object and $M \subset \mathbb{Q}$, then $A_M := \oplus_{i \in M} A_i$. We shall use this for $M \in \{\mathbb{N}, \mathbb{P}, \mathbb{Z}\}$. Similar notation is used in the bi-graded situation. This will be applied to various subspaces of $\mathfrak{g}$, the bi-grading being determined by $\mathbf{h}$. For instance, $\mathfrak{z}_{\mathfrak{g}}(\mathbf{e})$ is $\mathbb{Q}\times\mathbb{Q}$-graded and we may consider $\mathfrak{z}_{\mathfrak{g}}(\mathbf{e})_{\mathbb{P},\mathbb{P}}$. On the other hand, $\mathfrak{z}_{\mathfrak{g}}(e_2, h_1)$ lies in $\mathfrak{l}_1$ and hence is only $\mathbb{Q}$-graded. Therefore $\mathfrak{z}_{\mathfrak{g}}(e_2, h_1)_{\mathbb{P}}$ is defined. It is meant here that $\mathfrak{z}_{\mathfrak{g}}(e_2, h_1)_j$ is the subspace sitting in $\mathfrak{g}_{0,j} = (\mathfrak{l}_1)_j$.

Set $\mathfrak{h} := \mathfrak{z}_{\mathfrak{g}}(\mathbf{h}) = \mathfrak{g}_{0,0} = \mathfrak{l}_1 \cap \mathfrak{l}_2$ and consider the above filtrations and limits for $M = \mathfrak{h}$. It follows immediately form the definitions of the limits and Eq. (1.2) that

$$\textbf{(2.2)} \qquad \begin{aligned} &\lim_{e_1} \mathfrak{h} \subset \mathfrak{z}_{\mathfrak{g}}(e_1, h_2)_{\mathbb{P}} = \mathfrak{z}_{\mathfrak{l}_2}(e_1)_{\mathbb{P}} \\ &\lim_{e_2} \mathfrak{h} \subset \mathfrak{z}_{\mathfrak{g}}(e_2, h_1)_{\mathbb{P}} = \mathfrak{z}_{\mathfrak{l}_1}(e_2)_{\mathbb{P}} \\ &\lim_{\mathbf{e}} \mathfrak{h} \subset \mathfrak{z}_{\mathfrak{g}}(\mathbf{e})_{\mathbb{P},\mathbb{P}} \ . \end{aligned}$$

The next equalities follow from the fact that, by Lemma 2.1, all three spaces have the same dimension and the first two lie in the last one:

$$\textbf{(2.3)} \qquad \lim_{e_1}(\lim_{e_2} \mathfrak{h}) = \lim_{e_2}(\lim_{e_1} \mathfrak{h}) = \lim_{\mathbf{e}} \mathfrak{h} \ .$$

Now we are ready to introduce our main object.

2.4 Definition. A nilpotent pair $\mathbf{e}$ is called *wonderful* in $\mathfrak{g}$, if $\lim_{\mathbf{e}} \mathfrak{h} = \mathfrak{z}_{\mathfrak{g}}(\mathbf{e})_{\mathbb{P},\mathbb{P}}$; or, in a more explicit form, if

$$(\operatorname{ad} e_1)^i (\operatorname{ad} e_2)^j \mathfrak{h}(i, j) = \mathfrak{z}_{\mathfrak{g}}(\mathbf{e})_{i,j} \ \text{ for all } i, j \in \mathbb{P}.$$

Our primary goal is to describe properties of such pairs. For the sake of completeness, we start with an easy result.

2.5 Lemma. *Let* $\mathfrak{s}$ *be a semisimple subalgebra of* $\mathfrak{g}$ *and* $\mathbf{e} \in \mathfrak{s}\times\mathfrak{s}$ *a pair of commuting nilpotent elements. Then*

(i) $\mathbf{e}$ *is nilpotent in* $\mathfrak{s}$ *if and only if it is nilpotent in* $\mathfrak{g}$*;*

(ii) *if* $\mathbf{e}$ *is wonderful in* $\mathfrak{g}$, *then it is wonderful in* $\mathfrak{s}$.

Proof. (i) The 'only if' part is obvious.
Suppose $\mathbf{e}$ is nilpotent in $\mathfrak{g}$ and $(h_1, h_2) \in \mathfrak{g} \times \mathfrak{g}$ satisfies Eq. (1.2). Let $\mathfrak{g} = \mathfrak{s} \oplus \mathfrak{m}$ be an $\mathfrak{s}$-invariant decomposition and $h_i = h_i^{(\mathfrak{s})} + h_i^{(\mathfrak{m})}$, $i = 1, 2$. Then $(h_1^{(\mathfrak{s})}, h_2^{(\mathfrak{s})})$ also satisfy Eq. (1.2). This shows that $N_S := \{g \in S \mid g{\cdot}e_i \in \Bbbk e_i\}$ contains sufficiently many elements to ensure that 1.1(ii) holds for S in place of G.

(ii) Let $\mathbf{e}$ be wonderful in $\mathfrak{g}$. It follows from (i) and Theorem 1.4 that there exists a characteristic $\mathbf{h}$ lying in $\mathfrak{s} \times \mathfrak{s}$. The following is obvious. $\square$

2.6 Proposition. *Suppose* $\mathbf{e}$ *is wonderful. Then*

1. $\lim_{e_1} \mathfrak{h} = \mathfrak{z}_{\mathfrak{g}}(e_1, h_2)_{\mathbb{P}}$ *and* $\lim_{e_2} \mathfrak{h} = \mathfrak{z}_{\mathfrak{g}}(e_2, h_1)_{\mathbb{P}}$*;*
2. $\lim_{e_1} \mathfrak{z}_{\mathfrak{g}}(e_2, h_1, h_2) = \mathfrak{z}_{\mathfrak{g}}(e_2, e_1, h_2)_{\mathbb{P}}$ *and* $\lim_{e_2} \mathfrak{z}_{\mathfrak{g}}(e_1, h_1, h_2) = \mathfrak{z}_{\mathfrak{g}}(e_1, e_2, h_1)_{\mathbb{P}}$.

Proof. By symmetry, it is enough to prove one equality in each item.
1. Using (2.2) and (2.3) we obtain

$$\mathfrak{z}_{\mathfrak{g}}(\mathbf{e})_{\mathbb{P},\mathbb{P}} = \lim_{\mathbf{e}} \mathfrak{h} = \lim_{e_2}(\lim_{e_1} \mathfrak{h}) \subset \lim_{e_2}(\mathfrak{z}_{\mathfrak{g}}(e_1, h_2)_{\mathbb{P}}) \subset \mathfrak{z}_{\mathfrak{g}}(\mathbf{e})_{\mathbb{P},\mathbb{P}}.$$

2. Applying the formula in Definition 2.4 with $j = 0$ gives $(\mathrm{ad}\, e_1)^i \mathfrak{h}(i, 0) = \mathfrak{z}_{\mathfrak{g}}(\mathbf{e})_{i,0}$ for all $i \in \mathbb{P}$. Then summation over i yields the first formula. $\square$

The following simple lemma about $\mathbb{Z}$-graded Lie algebras appears to be extremely useful in our situation.

2.7 Lemma. *Let* $\mathfrak{a} = \oplus_{i \in \mathbb{Z}} \mathfrak{a}_i$ *be a* $\mathbb{Z}$*-graded Lie algebra. Suppose there exists* $e \in \mathfrak{a}_1$ *such that* $\dim \mathfrak{z}_{\mathfrak{a}}(e)_{\mathbb{P}} = \dim \mathfrak{a}_0$. *Then*

1. $\lim_e \mathfrak{a}_0 = \mathfrak{z}_{\mathfrak{a}}(e)_{\mathbb{P}}$ *and* $[\mathfrak{a}_i, e] = \mathfrak{a}_{i+1}$ *for all* $i \in \mathbb{P}$*;*
2. *If* $\mathfrak{a}$ *is reductive, then* $\mathfrak{z}_{\mathfrak{a}}(e) = \mathfrak{z}_{\mathfrak{a}}(e)_{\mathbb{P}}$ *and* e *is Richardson in the nilpotent radical of the parabolic subalgebra* $\mathfrak{a}_{\ge 0} := \oplus_{i \ge 0} \mathfrak{a}_i$.

Proof. 1. The first equality follows from the relations $\lim_e \mathfrak{a}_0 \subset \mathfrak{z}_{\mathfrak{a}}(e)_{\mathbb{P}}$ and $\dim \mathfrak{a}_0 = \dim(\lim_e \mathfrak{a}_0)$. The hypothesis on e also implies that the kernel of the map $(\mathrm{ad}\, e)_{\ge 0} : \mathfrak{a}_{\ge 0} \to \mathfrak{a}_{\ge 1}$ is of dimension $\dim \mathfrak{a}_0$. Thus $(\mathrm{ad}\, e)_{\ge 0}$ is onto.

2. Using an invariant nondegenerate bilinear form on $\mathfrak{a}$ and surjectivity of $(\mathrm{ad}\, e)_{\ge 0}$, one obtains $(\mathrm{ad}\, e)_{<0} : \mathfrak{a}_{\le -1} \to \mathfrak{a}_{\le 0}$ is injective. Hence $\mathfrak{z}_{\mathfrak{a}}(e)$ is concentrated in nonnegative degrees. The second claim is just a reformulation of the fact that $(\mathrm{ad}\, e)_{\ge 0}$ is onto. $\square$

Remark. If $\mathfrak{a}$ is reductive, the situation looks very much as if e were an even nilpotent element and the $\mathbb{Z}$-grading in question arose from an $\mathfrak{sl}_2$-triple containing e. This is not however always the case. For instance, consider $\mathfrak{a} = \mathfrak{sl}_n$. Here the $\mathbb{Z}$-grading with characteristic $(10\ldots0)$ satisfies the hypothesis in (2.7), but it does not correspond to an $\mathfrak{sl}_2$-triple. Moreover, the corresponding nilpotent element is not even.

2.8 Proposition. *If* $\mathbf{e}$ *is wonderful, then*

1. $\lim_{e_1} \mathfrak{h} = \mathfrak{z}_{\mathfrak{g}}(e_1, h_2)_{\mathbb{Z}} = \mathfrak{z}_{\mathfrak{g}}(e_1, h_2)_{\mathbb{P}}$ *and* $[\mathfrak{g}_{i,0}, e_1] = \mathfrak{g}_{i+1,0}$ *for all* $i \in \mathbb{P}$,
   $\lim_{e_2} \mathfrak{h} = \mathfrak{z}_{\mathfrak{g}}(e_2, h_1)_{\mathbb{Z}} = \mathfrak{z}_{\mathfrak{g}}(e_2, h_1)_{\mathbb{P}}$ *and* $[\mathfrak{g}_{0,j}, e_2] = \mathfrak{g}_{0,j+1}$ *for all* $j \in \mathbb{P}$*;*
2. $[\mathfrak{z}_{\mathfrak{g}}(e_2, h_2)_i, e_1] = \mathfrak{z}_{\mathfrak{g}}(e_2, h_2)_{i+1}$ *for all* $i \in \mathbb{P}$,
   $[\mathfrak{z}_{\mathfrak{g}}(e_1, h_1)_j, e_2] = \mathfrak{z}_{\mathfrak{g}}(e_1, h_1)_{j+1}$ *for all* $j \in \mathbb{P}$.

Proof. 1. In view of Proposition 2.6(1), the previous Lemma applies to reductive Lie algebras $(\mathfrak{l}_1)_{\mathbb{Z}}$ and $(\mathfrak{l}_2)_{\mathbb{Z}}$.
2. In view of Proposition 2.6(2), the previous Lemma applies to Lie algebras $\mathfrak{z}_{\mathfrak{g}}(e_1, h_1)_{\mathbb{Z}}$ and $\mathfrak{z}_{\mathfrak{g}}(e_2, h_2)_{\mathbb{Z}}$. $\square$

2.9 Theorem. *Let* $\mathbf{e}$ *and* $\mathbf{e}'$ *be two wonderful pairs with the same characteristic* $\mathbf{h}$*. Then there exists* $s \in H := Z_G(\mathbf{h})$ *such that* $s{\cdot}e_i = e_i'$, $i = 1, 2$.

Proof. We have $e_1, e_1' \in \mathfrak{g}_{1,0} \subset (\mathfrak{l}_2)_{\mathbb{Z}}$ and $e_2, e_2' \in \mathfrak{g}_{0,1} \subset (\mathfrak{l}_1)_{\mathbb{Z}}$. By proposition 2.8(1) with $i = 0$, $[\mathfrak{g}_{0,0}, e_1] = \mathfrak{g}_{1,0}$. This means that e_1 (and also e_1') lies in the dense H-orbit in $\mathfrak{g}_{1,0}$. Hence we may assume that $e_1 = e_1'$. Let H_{e_1} denote the stabilizer of e_1 in H. Then $\mathfrak{z}_{\mathfrak{g}}(e_1, h_1, h_2) = \mathfrak{z}_{\mathfrak{g}}(e_1, h_2)_0$ is Lie algebra of H_{e_1}. By proposition 2.8(2) with $j = 0$, $[\mathfrak{z}_{\mathfrak{g}}(e_1, h_1)_0, e_2] = \mathfrak{z}_{\mathfrak{g}}(e_1, h_1)_1$. This means that e_2 (and hence e_2') lies in the dense H_{e_1}-orbit in $\mathfrak{z}_{\mathfrak{g}}(e_1, h_1)_1$. Thus we can make $e_2 = e_2'$. $\square$

Next claim is a straightforward corollary of Theorems 1.6 and 2.9:

(2.10) *There exist finitely many* G*-orbits of wonderful pairs.*

Notice that the definition of wonderful pairs concerns only properties of integral eigenspaces of $\mathbf{h}$. There is therefore no harm in assuming that the eigenvalues of $\mathbf{h}$ in $\mathfrak{g}$ are integral. In this case, we say that $\mathbf{e}$ is *integral*.

2.11 Lemma. $\mathbf{e}$ *integral* $\Leftrightarrow$ $\mathfrak{z}_{\mathfrak{g}}(\mathbf{e}) = \mathfrak{z}_{\mathfrak{g}}(\mathbf{e})_{\mathbb{Z},\mathbb{Z}}$.

Proof. If (α, β) is a fractional eigenvalue of $\mathbf{h}$ in $\mathfrak{g}$, then, applying nilpotent endomorphisms $\mathrm{ad}\, e_1$ and $\mathrm{ad}\, e_2$ to $\mathfrak{g}_{\alpha,\beta}$, we eventually obtain an eigenvalue in $\mathfrak{z}_{\mathfrak{g}}(\mathbf{e})$ of the form $(\alpha + m, \beta + n)$ with $m, n \in \mathbb{P}$. $\square$

In general, the passage from $\mathfrak{g}$ to $\mathfrak{g}_{\mathbb{Z},\mathbb{Z}}$ can be considered as double analogue of taking the *even* part of the $\mathbb{Z}$-grading associated to an $\mathfrak{sl}_2$-triple. (A discrepancy is explained by the fact that, unlike Eq. (1.2), the standard normalization in $\mathfrak{sl}_2$-triples is: $[h, e] = 2e$.) For future references, we record the following fact which is a straightforward consequence of Lemma 2.7(2) and Proposition 2.8(1):

2.12 Lemma. *Let* $\mathbf{e}$ *be wonderful and integral. Then* e_1 *is Richardson in* $\mathfrak{l}_2$ *and* e_2 *is Richardson in* $\mathfrak{l}_1$.

The definition of a wonderful pair says something about $\mathfrak{z}_{\mathfrak{g}}(\mathbf{e})$ in the positive quadrant (of $\mathbb{Z}^2$-grading). This allows us to draw a conclusion about the negative quadrant.

2.13 Proposition. *Let* $\mathbf{e}$ *be wonderful and integral. Then* $\mathfrak{z}_{\mathfrak{g}}(\mathbf{e})_{p,q} = 0$ *whenever* $p < 0, q < 0$.

Proof. The argument used in the proof of Theorem 2.5(1) in [Pa99] applies here verbatim. For convenience of the reader, we reproduce it.

Assume $\mathfrak{z}_{\mathfrak{g}}(\mathbf{e})_{p,q}$ is nonzero for $p_0 = -p > 0$ and $q_0 = -q > 0$. It follows from the invariance of the Killing form on $\mathfrak{g}$ that $\mathfrak{z}_{\mathfrak{g}}(\mathbf{e})_{p,q} \neq 0$ if and only if $\mathfrak{g}_{p_0,q_0} \not\subset \mathrm{Im}\,(\mathrm{ad}\, e_1) + \mathrm{Im}\,(\mathrm{ad}\, e_2)$. By definition, put $\mathcal{D} = \mathfrak{g}_{p_0,q_0} \setminus (\mathrm{Im}\,(\mathrm{ad}\, e_1) + \mathrm{Im}\,(\mathrm{ad}\, e_2))$. For each $y \in \mathcal{D}$, consider the finite set $I_y = \{(k, l) \in (\mathbb{Z}_{\geq 0})^2 \mid (\mathrm{ad}\, e_1)^k (\mathrm{ad}\, e_2)^l y \neq 0\}$, with the lexicographic ordering. This means $(k, l) \prec (k', l') \Leftrightarrow k < k'$ or $k = k'$ and $l < l'$. Denote by $m(I_y)$ the unique maximal element in I_y. Let $y^* \in \mathcal{D}$ be an element such that $(k_0, l_0) := m(I_{y^*}) \preceq$

$m(I_z)$ for all $z \in \mathcal{D}$. Then $(\operatorname{ad} e_1)^{k_0}(\operatorname{ad} e_2)^{l_0} y^*$ is a nonzero element in $\mathfrak{z}_{\mathfrak{g}}(\mathbf{e}) \cap \mathfrak{g}_{p_0+k_0, q_0+l_0}$. By Definition 2.4, there is $t \in \mathfrak{h}(p_0+k_0, q_0+l_0)$ such that $(\operatorname{ad} e_1)^{p_0+k_0}(\operatorname{ad} e_2)^{q_0+l_0} t = (\operatorname{ad} e_1)^{k_0}(\operatorname{ad} e_2)^{l_0} y^*$. Then $(\operatorname{ad} e_1)^{k_0}(\operatorname{ad} e_2)^{l_0}(y^* - (\operatorname{ad} e_1)^{p_0}(\operatorname{ad} e_2)^{q_0} t) = 0$. Since $p_0 > 0, q_0 > 0$, we have $z^* = y^* - (\operatorname{ad} e_1)^{p_0}(\operatorname{ad} e_2)^{q_0} t$ is nonzero and belongs to $\mathcal{D}$. However, $I_{z^*} \subset I_{y^*} \setminus \{(k_0, l_0)\}$. Therefore $m(I_{z^*}) < m(I_{y^*})$, which contradicts the choice of y^*. Thus, the case $p<0, q<0$ is impossible. □

It appears that combining this proposition with information about $\mathfrak{z}_{\mathfrak{l}_1}(e_2)$ and $\mathfrak{z}_{\mathfrak{l}_2}(e_1)$ contained in 2.8, one obtains a characterization of the wonderful pairs.

2.14 Theorem. *Let* $\mathbf{e}$ *be integral. Then*

$$\lim_{\mathbf{e}} \mathfrak{h} = \mathfrak{z}_{\mathfrak{g}}(\mathbf{e})_{\mathbb{P},\mathbb{P}} \quad \Longleftrightarrow \quad \begin{cases} \mathfrak{z}_{\mathfrak{g}}(\mathbf{e})_{p,q} = 0 & \textit{for } \ p<0, q<0, \\ \mathfrak{z}_{\mathfrak{g}}(e_1)_{p,0} = 0 & \textit{for } \ p<0, \\ \mathfrak{z}_{\mathfrak{g}}(e_2)_{0,q} = 0 & \textit{for } \ q<0 \quad . \end{cases}$$

Proof. "$\Rightarrow$" This is Propositions 2.13 and 2.8(1).
"$\Leftarrow$" Making use of the Killing form on $\mathfrak{g}$, one can translate these three conditions in ones about the positive quadrant. Namely,

1st: $\mathfrak{g}_{p,q} \subset \operatorname{Im}(\operatorname{ad} e_1) + \operatorname{Im}(\operatorname{ad} e_2)$ for $p>0, q>0$;

2nd: $\mathfrak{g}_{p,0} \subset \operatorname{Im}(\operatorname{ad} e_1)$ for $p>0$;

3rd: $\mathfrak{g}_{0,q} \subset \operatorname{Im}(\operatorname{ad} e_2)$ for $q>0$.

These three together show that $\mathfrak{g}_{0,0} (= \mathfrak{h})$, e_1, and e_2 generate $\mathfrak{g}_{\mathbb{P},\mathbb{P}}$. In particular, applying $\operatorname{ad} e_1$ and $\operatorname{ad} e_2$ to $\mathfrak{h}$, we obtain the whole space $\mathfrak{z}_{\mathfrak{g}}(\mathbf{e})_{\mathbb{P},\mathbb{P}}$. □

Of course, all statements about integral wonderful pairs can be reformulated as ones about integral eigenspaces of arbitrary wonderful pairs.

3 Classes of wonderful pairs

First of all, notice that a rectangular nilpotent pair is wonderful. This is an exercise for the reader. Next, the principal and almost principal pairs are wonderful. This follows from [Gi99, sect. 1] for the former and from [Pa99, 2.3] for the latter. As we have now the general concept of a characteristic of a nilpotent pair, it is worth to put these notions in a more general context.

Recall that the even nilpotent elements in $\mathfrak{g}$ are characterized by the following property:

Let $\{e, \tilde{h}, f\}$ be an $\mathfrak{sl}_2$-triple. Then e is even if and only if $\dim \mathfrak{z}_{\mathfrak{g}}(e) = \dim \mathfrak{z}_{\mathfrak{g}}(\tilde{h})$.

Translating this into the double setting, we shall say that a nilpotent pair $\mathbf{e}$ is

- *even* whenever $\dim \mathfrak{z}_{\mathfrak{g}}(\mathbf{e}) = \dim \mathfrak{z}_{\mathfrak{g}}(\mathbf{h})$;
- *almost even* whenever $\dim \mathfrak{z}_{\mathfrak{g}}(\mathbf{e}) = \dim \mathfrak{z}_{\mathfrak{g}}(\mathbf{h}) + 1$.

Notice that the second assumption implies that $\mathbf{e}$ is non-trivial, because $\dim \mathfrak{z}_{\mathfrak{g}}(e) - \dim \mathfrak{z}_{\mathfrak{g}}(h)$ is always even for the sole nilpotent element. Since $\operatorname{rk} \mathfrak{g} \le \dim \mathfrak{z}_{\mathfrak{g}}(\mathbf{h}) \le$

$\dim \mathfrak{z}_{\mathfrak{g}}(\mathbf{e})$ and $\dim \mathfrak{z}_{\mathfrak{g}}(\mathbf{h}) - \mathrm{rk}\, \mathfrak{g}$ is even, any principal pair is even and almost principal pair is almost even. Our terminology is partly justified by the following observation.

3.1 Proposition. *Let $\mathbf{e}$ be a rectangular nilpotent pair. Then $\mathbf{e}$ is even if and only if both e_1 and e_2 are even nilpotent elements.*

Proof. Let $\{e_1, \tilde{h}_1, f_1\}$ and $\{e_2, \tilde{h}_2, f_2\}$ be commuting $\mathfrak{sl}_2$-triples and let $\mathfrak{g} = \oplus_{i\in\mathbb{Z}}\mathfrak{g}(i)$ be the grading determined by $\tilde{h}_1$. Then $e_1 \in \mathfrak{g}(2)$ and $e_2 \in \mathfrak{g}(0)$. Set $\mathfrak{k}_1 = \mathfrak{z}_{\mathfrak{g}}(e_1, \tilde{h}_1, f_1)$. Recall that $(\tilde{h}_1, \tilde{h}_2) = 2\mathbf{h}$ in the rectangular case. It is easily seen that $\mathfrak{z}_{\mathfrak{g}}(e_1)$ and $\mathfrak{g}(0) \oplus \mathfrak{g}(1)$ are isomorphic $\mathfrak{k}_1$-modules. (In the notation of sect. 2, $\lim_{e_1}(\mathfrak{g}(0) \oplus \mathfrak{g}(1)) = \mathfrak{z}_{\mathfrak{g}}(e_1)$.) As $e_2 \in \mathfrak{k}_1$, we obtain

$$\dim \mathfrak{z}_{\mathfrak{g}}(e_1, e_2) = \dim \mathfrak{z}_{\mathfrak{g}(0)}(e_2) + \dim \mathfrak{z}_{\mathfrak{g}(1)}(e_2) \ .$$

Since $\dim \mathfrak{z}_{\mathfrak{g}}(h_1, h_2) = \dim \mathfrak{z}_{\mathfrak{g}(0)}(h_2) \le \dim \mathfrak{z}_{\mathfrak{g}(0)}(e_2)$, we see that $\mathbf{e}$ is even if and only if $\mathfrak{g}(1) = 0$ (i.e. e_1 is even) and e_2 is even in $\mathfrak{g}(0)$. Then either by symmetry or by a direct argument one concludes that e_2 is actually even in $\mathfrak{g}$. □

Using the notation of the previous proof, it is easy to state the similar condition for $\mathbf{e}$ being almost even.

3.2 Proposition. *A rectangular nilpotent pair (e_1, e_2) is almost even if and only if e_2 is even in $\mathfrak{g}(0)$ and $\mathfrak{g}(1)$ is an irreducible $\langle e_2, \tilde{h}_2, f_2\rangle$-module.*

Proof. Left to the reader. □

It turns out that many statements about principal and almost principal pairs proved in [Gi99] and [Pa99] remain true, with essentially the same proofs, for the even and almost even pairs. Not trying to be exhaustive in this rewriting, we demonstrate several results. Recall that $\mathfrak{h} = \mathfrak{z}_{\mathfrak{g}}(\mathbf{h})$.

3.3 Theorem. 1. *The following conditions are equivalent:*

(i) $\lim_{\mathbf{e}} \mathfrak{h} = \mathfrak{z}_{\mathfrak{g}}(\mathbf{e})$,

(ii) $\mathfrak{z}_{\mathfrak{g}}(\mathbf{e})_{\mathbb{P},\mathbb{P}} = \mathfrak{z}_{\mathfrak{g}}(\mathbf{e})$;

2. *Any even nilpotent pair is wonderful and integral.*

Proof. 1. $(i) \Rightarrow (ii)$ – We always have $\lim_{\mathbf{e}} \mathfrak{h} \subset \mathfrak{z}_{\mathfrak{g}}(\mathbf{e})_{\mathbb{P},\mathbb{P}} \subset \mathfrak{z}_{\mathfrak{g}}(\mathbf{e})$.
$(ii) \Rightarrow (i)$. As the centralizer of $\mathbf{e}$ entirely lies in the positive quadrant, we have $(\mathrm{ad}\, e_1)_{p,q} : \mathfrak{g}_{p,q} \to \mathfrak{g}_{p+1,q}$ is injective for all q and $p < 0$. (Otherwise, applying $\mathrm{ad}\, e_2$ to a nonzero element in the kernel we would eventually arrived at an element in $\mathfrak{z}_{\mathfrak{g}}(\mathbf{e})_{p,q'}$ with $q' \ge q$.) Similarly, $(\mathrm{ad}\, e_2)_{p,q} : \mathfrak{g}_{p,q} \to \mathfrak{g}_{p,q+1}$ is injective for all p and $q < 0$. We thus have more than enough to apply Theorem 2.14 and conclude that $\lim_{\mathbf{e}} \mathfrak{h} = \mathfrak{z}_{\mathfrak{g}}(\mathbf{e})_{\mathbb{P},\mathbb{P}}$.

2. This follows from the first part and from Lemma 2.11. □

The following sufficient condition will be helpful in our study of almost even nilpotent pairs.

3.4 Proposition. *Let* $\mathbf{e}$ *be a nilpotent pair in* $\mathfrak{g}$. *Suppose* $\dim \mathbb{V}^{\langle e_1,e_2\rangle} = 1$ *for a self-dual* $\mathfrak{g}$*-module* $\mathbb{V}$. *Then* $\mathbf{e}$ *is a rectangular principal nilpotent pair in* $\mathfrak{g}$.

Proof. We first prove that $\mathbf{e}$ has prescribed properties as nilpotent pair in $\mathfrak{sl}(\mathbb{V})$ and then go down to $\mathfrak{g}$.

1. Take a characteristic $\mathbf{h}$ of $\mathbf{e}$ and consider the corresponding bi-grading $\mathbb{V} = \oplus_{p,q\in\mathbb{Q}} \mathbb{V}_{p,q}$. Set $\Gamma = \{(p,q) \mid \mathbb{V}_{p,q} \neq 0\} \subset \mathbb{Q}\times\mathbb{Q}$. Each nonempty coset $((p',q') + (\mathbb{Z}\times\mathbb{Z})) \cap \Gamma$ determines a subspace in $\mathbb{V}$ containing a $\langle e_1,e_2\rangle$-fixed vector (cf. 2.11). Hence Γ lies in a unique such coset. For the same reason, $\dim \mathbb{V}_{p,q} = 1$ for all $(p,q)\in\Gamma$, and Γ has a unique 'northeast' corner, i.e. $\#\{(p,q)\in\Gamma \mid (p{+}1,q)\notin\Gamma \ \&\ (p,q{+}1)\notin\Gamma\} = 1$. Let (p_0,q_0) be this corner. Since $\mathbb{V}$ is self-dual, Γ is centrally-symmetric. Whence $(-p_0,-q_0)$ is the unique 'southwest' corner of it. (Note that, although p_0,q_0 are not necessarily integral, $(p_0,q_0)\in(-p_0,-q_0)+(\mathbb{Z}\times\mathbb{Z})$ implies $p_0,q_0\in\frac{1}{2}\mathbb{Z}$.) It follows that Γ lies inside of the rectangle having opposite vertices (p_0,q_0) and $(-p_0,-q_0)$. It is however easy to see that the conditions $[e_1,e_2]=0$ and $\dim \mathbb{V}^{\langle e_1,e_2\rangle}=1$ force that Γ is "equal" to this rectangle, i.e., $\Gamma = \{(m,n) \mid p_0-m\in\mathbb{Z}, q_0-n\in\mathbb{Z}, |m|\le p_0, |n|\le q_0\}$.

2. Since the eigenvalues of $\mathbf{h}$ in $\mathbb{V}$ form a rectangle, $\mathbf{e}$ is a rectangular principal nilpotent pair in $\mathfrak{sl}(\mathbb{V})$. Indeed, it is not hard to write a formula for the nilpotent operators f_i such that $[e_j,f_i] = \delta_{i,j}2h_j$ and $[h_j,f_i] = -\delta_{i,j}f_i$ $(i,j\in\{1,2\})$. Hence $\mathbf{e}$ is rectangular. Set $\mathfrak{a}_i = \langle e_i,h_i,f_i\rangle$. As $\mathfrak{gl}(\mathbb{V}) \simeq \mathbb{V}\otimes\mathbb{V}^*$ and $\mathbb{V}$ is an irreducible $\mathfrak{a}_1\times\mathfrak{a}_2$-module, an explicit computation shows that $\dim \mathfrak{gl}(\mathbb{V})^{\langle e_1,e_2\rangle} = \dim\mathbb{V} = \mathrm{rk}\,\mathfrak{gl}(\mathbb{V})$ (cf. [EP99]). Hence $\mathbf{e}$ is principal in $\mathfrak{sl}(\mathbb{V})$.

3. Now, one has the following: $\mathfrak{g}\subset\mathfrak{sl}(\mathbb{V})$ is semisimple, $\{e_i,2h_i,f_i\}$ is an $\mathfrak{sl}_2$-triple in $\mathfrak{sl}(\mathbb{V})$, and $e_i,2h_i\in\mathfrak{g}$. This easily implies that $f_i\in\mathfrak{g}$ and $\mathbf{e}$ is rectangular in $\mathfrak{g}$.

4. It follows from the general theory of principal nilpotent pairs [Gi99, sect. 1] and is also easily seen in our situation that $\mathfrak{z}_{\mathfrak{sl}(\mathbb{V})}(\mathbf{h})$ is a Cartan subalgebra (the set of diagonal traceless matrices). Whence $\mathfrak{z}_{\mathfrak{g}}(\mathbf{h})$ is a Cartan subalgebra in $\mathfrak{g}$. Because the eigenvalues of $2h_i$ in $\mathfrak{g}$ are even $(i=1,2)$, all irreducible $\mathfrak{a}_1\times\mathfrak{a}_2$-submodules in $\mathfrak{sl}(\mathbb{V})$, and hence in $\mathfrak{g}$, have zero weight. Thus $\dim\mathfrak{z}_{\mathfrak{g}}(\mathbf{h}) = \dim\mathfrak{z}_{\mathfrak{g}}(\mathbf{e})$ and $\mathbf{e}$ is principal in $\mathfrak{g}$. □

3.5 Remarks. 1. It follows from the proof that $\mathbb{V}$ is an irreducible $\mathfrak{a}_1\times\mathfrak{a}_2$-module and, (p_0,q_0) being the eigenvalue of $\mathbf{h}$ on $\mathbb{V}^{\langle e_1,e_2\rangle}$, $\dim\mathbb{V} = (2p_0+1)(2q_0+1)$.
2. Since $\mathbb{V}$ is assumed to be self-dual, $\mathfrak{g}$ lies in either $\mathfrak{sp}(\mathbb{V})$ or $\mathfrak{so}(\mathbb{V})$. Obviously, $\mathfrak{g}\hookrightarrow\mathfrak{so}(\mathbb{V})$ if and only if $\mathbb{V}$ is an orthogonal $\mathfrak{a}_1\times\mathfrak{a}_2$-module if and only if $p_0-q_0\in\mathbb{Z}$. In any case, $\mathbf{e}$ is principal in the respective classical Lie algebra. Furthermore, it is not hard to list all possibilities for such $(\mathfrak{g},\mathbb{V})$, see [ElPa].

3.6 Theorem. *Suppose* $\mathbf{e}$ *is almost even. Then* $\mathfrak{z}_{\mathfrak{g}}(\mathbf{e}) = \lim_{\mathbf{e}}\mathfrak{h}\oplus\langle x\rangle$, *where* $x\in\mathfrak{g}_{p,q}$.
1. *There are two possibilities for the eigenvalue of* x: *either* $p,q\in\mathbb{Z}$ *and* $pq<0$ *or* $p,q\in\frac{1}{2}\mathbb{Z}\setminus\mathbb{Z}$ *and* $p>0, q>0$. *In particular,* $\mathbf{e}$ *is wonderful.*
2. *In case* (p,q) *is fractional, we have*

$\mathfrak{h}$ *is a Cartan subalgebra in* $\mathfrak{g}$,
$\mathbf{e}$ *is an almost principal rectangular pair,*
$\mathbf{e}$ *is principal in* $\mathfrak{g}_{\mathbb{Z},\mathbb{Z}}$.

Proof. Since $\lim_{\mathbf{e}}\mathfrak{h}$ is $\mathbf{h}$-stable, the first equality follows for dimension reason.
1. (a) Suppose (p,q) is integral. That the case $p\ge 0, q\ge 0$ is impossible follows from Theorem 3.3(1). This already means that $\mathbf{e}$ is wonderful. Consequently, results of sect. 2 apply. By Theorem 2.14, neither $p<0, q<0$ nor $pq=0$ can occur. We are thus left with the case $pq<0$.

(b) Suppose (p,q) is fractional. Let $\mathfrak{g}_{fr}$ be the sum of all fractional eigenspaces. Then $\mathfrak{g}_{fr}$ is an orthogonal $\mathfrak{g}_{\mathbb{Z},\mathbb{Z}}$-module, $\mathfrak{g}=\mathfrak{g}_{\mathbb{Z},\mathbb{Z}}\oplus\mathfrak{g}_{fr}$, and $\mathfrak{g}_{fr}^{\langle e_1,e_2\rangle}=\langle x\rangle$. Applying Proposition 3.4, we conclude that $p,q\in\frac{1}{2}\mathbb{Z}$, $\mathbf{e}$ is rectangular principal in $\mathfrak{g}_{\mathbb{Z},\mathbb{Z}}$, and $\mathfrak{z}_{\mathfrak{g}}(\mathbf{h})$ is a Cartan subalgebra of $\mathfrak{g}$. Moreover, it follows from the orthogonality that both p,q must be fractional (cf. Remark 3.5). Finally, $\dim\mathfrak{z}_{\mathfrak{g}}(\mathbf{e})=\operatorname{rk}\mathfrak{g}_{\mathbb{Z},\mathbb{Z}}+1=\operatorname{rk}\mathfrak{g}+1$, i.e., $\mathbf{e}$ is almost principal in $\mathfrak{g}$. $\square$

Remark. Notice that in this case the decomposition $\mathfrak{g}=\mathfrak{g}_{\mathbb{Z},\mathbb{Z}}\oplus\mathfrak{g}_{fr}$ is a $\mathbb{Z}_2$-grading and the corresponding involutory automorphism of $\mathfrak{g}$ is inner.

3.7 Example. If $\mathbf{e}$ is rectangular and $\mathbf{h}$ is a characteristic of it, then the eigenvalues of h_i belong to $\frac{1}{2}\mathbb{Z}$ $(i=1,2)$. It was also shown above that for the even nilpotent pairs the eigenvalues of $\mathbf{h}$ are integral. There exist however wonderful pairs whose eigenvalues have arbitrarily large denominators. An example can be constructed as follows. Let $\mathbf{e}$ be a principal nilpotent pair in $\mathfrak{sl}_n$ corresponding to the partition $(n-1,1)$. (See [Gi99, sect. 5] for the relationship between principal nilpotent pairs in $\mathfrak{sl}_n$ and partitions.) Consider $\mathbf{e}$ as nilpotent pair in $\mathfrak{sp}_{2n}$, using the natural inclusion $\mathfrak{sl}_n\hookrightarrow\mathfrak{sp}_{2n}$. Then $\mathbf{e}$ is a non-integral wonderful pair in $\mathfrak{sp}_{2n}$ and eigenvalues of $\mathbf{h}$ in $\mathfrak{sp}_{2n}=\mathfrak{g}$ have denominator 'n'. For simplicity, take $n=3$. Having chosen a Witt basis in $\Bbbk^6$ for the symplectic form, we may take $e_1=v_{23}-v_{45}$ and $e_2=v_{13}-v_{46}$, where $\{v_{ij}\}$ is the monomial basis in the space of matrices. Then $h_1=\frac{1}{3}\mathrm{diag}(-1,2,-1,1,-2,1)$, $h_2=\frac{1}{3}\mathrm{diag}(2,-1,-1,1,1,-2)$. Here $\dim\mathfrak{z}_{\mathfrak{g}}(\mathbf{e})=7$ and the eigenvalues of $\mathbf{h}$ on $\mathfrak{z}_{\mathfrak{g}}(\mathbf{e})$ are $(0,0)$, $(1,0)$, $(0,1)$, $(1/3,1/3)$, $(2/3,2/3)$, $(4/3,-2/3)$, $(2/3,-4/3)$.

Thus, there is no universal bound for denominators of the eigenvalues of $\mathbf{h}$. Yet, one can give a (very rough) estimate for each $\mathfrak{g}$, which actually applies to characteristics of arbitrary nilpotent pairs. If $\mathfrak{s}$ is a semisimple Lie algebra, let $c(\mathfrak{s})$ denote the determinant of the Cartan matrix of $\mathfrak{s}$ or, equivalently, the order of the centre of the corresponding simply-connected group.

3.8 Lemma. *Let* $h\in\mathfrak{s}$ *be a semisimple element. Suppose the eigenvalues of* $\operatorname{ad}h$ *are integral. For any finite-dimensional representation* $\rho:\mathfrak{s}\to\mathfrak{sl}(\mathbb{V})$, *then the eigenvalues of* $\rho(h)$ *belong to* $\frac{1}{c(\mathfrak{s})}\mathbb{Z}$.

Proof. The rows of the inverse of the Cartan matrix yield the expressions of the fundamental weights through the simple roots. □

3.9 Proposition. *Let* $\mathbf{h}$ *be a characteristic of a nilpotent pair in* $\mathfrak{g}$. *Then the denominators of the eigenvalues of* $\operatorname{ad} h_1$, $\operatorname{ad} h_2$ *do not exceed* $\max\limits_{\mathfrak{s}\subset\mathfrak{g}} c(\mathfrak{s})$, *where* $\mathfrak{s}$ *ranges over all regular semisimple subalgebras of* $\mathfrak{g}$.

Proof. As in (3.6), consider the decomposition $\mathfrak{g} = \mathfrak{g}_{\mathbb{Z},\mathbb{Z}} \oplus \mathfrak{g}_{fr}$ and set $\mathfrak{s} = [\mathfrak{g}_{\mathbb{Z},\mathbb{Z}}, \mathfrak{g}_{\mathbb{Z},\mathbb{Z}}]$. Then $h_1, h_2 \in \mathfrak{s}$ and $\mathfrak{g}_{fr}$ is an $\mathfrak{s}$-module. Now we conclude by the previous lemma. □

However I think that for wonderful pairs a better estimate ought to exist.

4 Characteristics for principal and almost principal pairs

We give a version of Theorem 1.7 for principal and almost principal integral pairs[1]. When we shall give two references for a property of such pairs, this means that the proof is found in [Gi99] for principal pairs and in [Pa99] for almost principal ones.

4.1 Lemma. *Let* $\mathbf{e}$ *be either a principal or an almost principal integral pair. Then* $\mathbf{e}$ *is not contained in a proper regular semisimple subalgebra of* $\mathfrak{g}$.

Proof. (Cf. Remark after 4.4 in [Pa99].) Assume that $\mathbf{e}$ is contained in a proper regular semisimple subalgebra and let $\tilde{\mathfrak{g}}$ be a maximal one among them. It follows from [Dy52a, §5] and V. Kac's description of periodic automorphisms of $\mathfrak{g}$ (see e.g. [VGO90, 3.7]) that $\tilde{\mathfrak{g}}$ is a fixed-point subalgebra of some *inner* automorphism of $\mathfrak{g}$ of finite order. Since $e_1, e_2 \in \tilde{\mathfrak{g}}$, this means $Z_G(\mathbf{e})$ contains non-trivial semisimple elements. But $Z_G(\mathbf{e})$ is connected unipotent for such $\mathbf{e}$ (see [Gi99, 3.6] and [Pa99, 2.14]). □

By Ginzburg's result [Gi99, 3.4], both $G{\cdot}e_1$ and $G{\cdot}e_2$ are Richardson orbits in $\mathfrak{g}$, if $\mathbf{e}$ is principal. But in the almost principal case only one of them is Richardson [Pa99, 2.9(ii), 2.10]. In either case, if the orbit $G{\cdot}e_2$ is Richardson, a more precise statement is: Set $\mathfrak{p}_2 = \bigoplus\limits_{i\in\mathbb{Z}, j\in\mathbb{P}} \mathfrak{g}_{i,j} = \mathfrak{g}_{*,\mathbb{P}}$. It is a parabolic subalgebra and $\mathfrak{l}_2$ is a Levi subalgebra in it. Then $e_2 \in (\mathfrak{p}_2)^{nil} = \mathfrak{g}_{*,\mathbb{N}}$ and $[\mathfrak{p}_2, e_2] = (\mathfrak{p}_2)^{nil}$.

For a reductive Lie algebra $\mathfrak{l}$, let $\operatorname{Cxt}(\mathfrak{l})$ denote maximum among the Coxeter numbers of the simple components of $\mathfrak{l}$. Recall that $\mathfrak{z}_{\mathfrak{g}}(\mathbf{h})$ is a Cartan subalgebra in the principal and almost principal case.

4.2 Theorem. *Let* $\mathbf{e}$ *be either a principal or an almost principal integral pair. Choose the set of simple roots* Π *relative to* $\mathfrak{z}_{\mathfrak{g}}(\mathbf{h})$ *so that* $h_2 + \varepsilon h_1$ *is strictly dominant for all sufficiently small* $\varepsilon \in \mathbb{Q}$. *Then*

(i) $\alpha(h_2) \in \{0, 1\}$ *for all* $\alpha \in \Pi$;

(ii) *If* $\alpha(h_2) = 0$, *then* $\alpha(h_1) = 1$;

[1] in [Pa99], we used the term "pairs of $\mathbb{Z}$-type" for integral pairs.

(iii) *If* $\alpha(h_2)=1$ *and* $G{\cdot}e_2$ *is Richardson, then* $\alpha(h_1)\in\{-\mathrm{Cxt}(\mathfrak{l}_2){+}1,\dots,-1,0\}$.

Proof. Using Theorem 1.7, Lemma 4.1, and the fact that $\mathfrak{z}_{\mathfrak{g}}(\mathbf{h})$ is Cartan (hence the case $\alpha(h_1)=\alpha(h_2)=0$ is impossible), one sees that we have to only prove that $\alpha(h_1)\ge -\mathrm{Cxt}(\mathfrak{l}_2){+}1$. By [Gi99, sect. 1] and [Pa99, 2.3], e_1 is regular nilpotent in $\mathfrak{l}_2$. It then follows from (ii) that the $\mathbb{Z}$-grading in $\mathfrak{l}_2$ defined by h_1 is nothing but the standard grading associated with the function $\alpha\mapsto \mathrm{ht}\,(\alpha)$ on the set of roots of $\mathfrak{l}_2$ (i.e., $(\mathfrak{l}_2)_i$ is the linear span of the root spaces such that the height of the corresponding root of $\mathfrak{l}_2$ is i). Therefore $\min\{i\mid \mathfrak{g}_{i,0}\ne 0\}=-\mathrm{Cxt}(\mathfrak{l}_2){+}1$. If e_2 is Richardson in $(\mathfrak{p}_2)^{nil}$, then $[\mathfrak{g}_{*,0},e_2]=\mathfrak{g}_{*,1}$. Hence $\min\{i\mid \mathfrak{g}_{i,1}\ne 0\}\ge -\mathrm{Cxt}(\mathfrak{l}_2){+}1$, which is exactly what we need. □

Recall that the exponents of a simple Lie algebra are the degrees of fundamental polynomial invariants of the adjoint representation, reduced by 1. The set of exponents of a reductive Lie algebra is the union of the exponents of all simple components. Since the sum of the exponents is the dimension of a maximal nilpotent subalgebra, the following is a generalization of [Gi99, 6.13].

4.3 Theorem. *Let* $\mathbf{e}$ *be either principal or almost principal and let* (α_i,β_i) $(i=1,\dots,\mathrm{rk}\,\mathfrak{g})$ *be the eigenvalues of* $\mathbf{h}$ *in* $\lim_{\mathbf{e}}\mathfrak{h}=\mathfrak{z}_{\mathfrak{g}}(\mathbf{e})_{\mathbb{P},\mathbb{P}}$. *Then* $\{\alpha_i\mid\beta_i\ne 0\}$ *are the exponents of* $\mathfrak{l}_2$ *and* $\{\beta_i\mid\alpha_i\ne 0\}$ *are the exponents of* $\mathfrak{l}_1$.

Proof. 1. First, assume that $\mathbf{e}$ is either principal or almost principal integral.
Since $\mathbf{e}$ is wonderful and integral in both cases, the formulas in (2.8) become simpler. In particular, $\lim_{e_1}\mathfrak{h}=\mathfrak{z}_{\mathfrak{g}}(e_1,h_2)=\mathfrak{z}_{\mathfrak{l}_2}(e_1)$. Using (2.3), we obtain $\lim_{\mathbf{e}}\mathfrak{h}=\lim_{e_2}\mathfrak{z}_{\mathfrak{l}_2}(e_1)$. Because $[h_1,e_2]=0$, this implies that the eigenvalues of h_1 in $\lim_{\mathbf{e}}\mathfrak{h}$ and $\mathfrak{z}_{\mathfrak{l}_2}(e_1)$ are the same. Therefore $\{\alpha_i\mid\beta_i\ne 0\}$ are just the eigenvalues of h_1 in $\mathfrak{z}_{\mathfrak{l}_2}(e_1)_{\mathbb{N}}$. Since $[e_1,(\mathfrak{l}_2)_i]=(\mathfrak{l}_2)_{i+1}$ for $i=1,2,\dots$ (see 2.8(1)), the partition dual to $(\dim(\mathfrak{l}_2)_1,\dim(\mathfrak{l}_2)_2,\dots)$ consists of the h_1-eigenvalues in $\mathfrak{z}_{\mathfrak{l}_2}(e_1)_{\mathbb{N}}$. But, since the $\mathbb{Z}$-grading of $\mathfrak{l}_2$ is the standard grading associated with the height of roots (see the proof of 4.2), the dual partition consists also of the exponents of $\mathfrak{l}_2$. This is a classical result of Kostant [Ko59], see also [CM93, ch. 4]. This argument is completely symmetric with respect to e_1 and e_2, because we do not need the assumption (in the almost principal case) that e_2 is Richardson.
2. Assume that $\mathbf{e}$ is almost principal non-integral. Then $\mathbf{e}$ is principal in $\mathfrak{g}_{\mathbb{Z},\mathbb{Z}}$ (see [Pa99, 2.7] or 3.6(ii)) and we conclude by the first part of the proof. □

(4.4) Examples. 1. $\mathfrak{g}=\mathfrak{e}_6$. According to [EP99, 7.6], there is a principal nilpotent pair $\mathbf{e}$ such that $G{\cdot}e_1$ is of type $\mathbf{D}_5$ and $G{\cdot}e_2$ is of type $2\mathbf{A}_1$. This means that e_1 (resp. e_2) is regular in some Levi subalgebra of type $\mathbf{D}_5$ (resp. $2\mathbf{A}_1$). We are going to write down explicitly h_1 and h_2 for this nilpotent pair. Choosing the set of simple roots as in Theorem 4.2, we see that h_2 is dominant and $\mathfrak{l}_2$ has to be a standard Levi subalgebra of type $\mathbf{D}_5$. Up to the symmetry of Dynkin diagram, there is a unique possibility for this: $h_2=\begin{pmatrix}0\ 0\ 0\ 0\ 1\\ 0\end{pmatrix}$. Then $h_1=\begin{pmatrix}1\ 1\ 1\ 1\ x\\ 1\end{pmatrix}$, where $-\mathrm{Cxt}(\mathbf{D}_5){+}1=-7\le x\le 0$. We identify

h_i with the collection $\alpha_j(h_i)$, $1 \le j \le 6$. Making use of Ginzburg's results [Gi99, sect. 6] and some 'ad hoc' arguments, one concludes that (h_1, h_2) can be a characteristic of a principal pair if and only if $x = -7$. The corresponding $\mathbb{Z}^2$-grading of $\mathfrak{e}_6$ is depicted in Figure 1.

Figure 1:

$$
\begin{array}{cccccccccccccccccc}
 & & & & & & & & \uparrow^{h_2} & & & & & & & & \\
 & 1 & 1 & 1 & 2 & 2 & 2 & 2 & 2^* & 1 & 1 & 1^* & & & & & \\
- & 1 & 1 & 2 & 3 & 4 & 4 & 5 & 6 & 5^* & 4 & 4 & 3^* & 2^* & 1 & 1^* & \xrightarrow{h_1} \\
 & & & & & 1 & 1 & 1 & 2 & 2 & 2 & 2 & 2 & 1 & 1 & 1 & \\
 & & & & & & & & | & & & & & & & &
\end{array}
$$

This means, for instance, that $\dim \mathfrak{g}_{0,0} = 6$, $\dim \mathfrak{g}_{-4,1} = 2$ and $\dim \mathfrak{g}_{1,0} = 5$. The superscript '*' refers to the eigenspaces containing a one-dimensional space from $\mathfrak{z}_{\mathfrak{g}}(\mathbf{e})$.

2. $\mathfrak{g} = \mathfrak{e}_7$. According to [EP99], there is a principal nilpotent pair $\mathbf{e}$ such that both $G{\cdot}e_1$ and $G{\cdot}e_2$ are of type $\mathbf{A}_4 + \mathbf{A}_1$. Here one has several possibilities for standard Levi subalgebras of type $\mathbf{A}_4 + \mathbf{A}_1$. Using properties of principal pairs and case-by-case arguments, one finds that
$h_1 = \left(\begin{smallmatrix} -1 & 1 & -4 & 1 & 1 & 1 \\ & & 1 & & & \end{smallmatrix} \right)$ and $h_2 = \left(\begin{smallmatrix} 1 & 0 & 1 & 0 & 0 & 0 \\ & & 0 & & & \end{smallmatrix} \right)$. This yields an extremely beautiful $\mathbb{Z}^2$-grading of $\mathfrak{e}_7$, see Figure 2.

Figure 2:

$$
\begin{array}{ccccccccccc}
 & & & & & \uparrow^{h_2} & & & & & \\
 & 1 & 1 & 1 & 1 & 1^* & & & & & \\
 & 1 & 2 & 2 & 2 & 2 & 1^* & & & & \\
 & 1 & 2 & 3 & 3 & 3 & 2 & 1^* & & & \\
 & 1 & 2 & 3 & 5 & 5^* & 3 & 2 & 1^* & & \\
- & 1 & 2 & 3 & 5 & 7 & 5^* & 3 & 2 & 1^* & \xrightarrow{h_1} \\
 & & 1 & 2 & 3 & 5 & 5 & 3 & 2 & 1 & \\
 & & & 1 & 2 & 3 & 3 & 3 & 2 & 1 & \\
 & & & & 1 & 2 & 2 & 2 & 2 & 1 & \\
 & & & & & 1 & 1 & 1 & 1 & 1 & \\
 & & & & & | & & & & &
\end{array}
$$

3. In [Pa99, 2.15], we described a series of integral almost principal pairs in $\mathfrak{sp}_{4n}$. The orbit $G{\cdot}e_1$ corresponds to the partition $(2n, 2n)$ and is Richardson whereas $G{\cdot}e_2$ corresponds to $(2^{2n-1}, 1, 1)$ and is not Richardson. The formulas for $\mathbf{h}$ imply that $\mathfrak{l}_2 = \mathbf{A}_{2n-1}$ and $\mathfrak{l}_1 = n\mathbf{A}_1$. These formulas are such that neither of h_i is dominant with respect to the standard set of simple roots. Choosing Π adapted to h_2, as in Theorem 4.2, we obtain $h_1 = (1\ 1 \ldots 1\ \ -2n)$, $h_2 = (0\ 0 \ldots 0\ 1)$. Since $-2n < -2n+1 = -\mathrm{Cxt}(\mathbf{A}_{2n-1})+1$, we see that 4.2(iii) does not hold here. But choosing Π adapted to h_1, we obtain $h_1 = (1\,0 \ldots 1\,0)$, $h_2 = (-1\,1 \ldots -1\,1)$, which correlates with the fact that $\mathrm{Cxt}(\mathbf{A}_1) = 2$. This shows the condition of being Richardson for $G{\cdot}e_2$ is essential in 4.2(iii).

(4.5) Classification problems. A usual intention of a mathematician is to classify something, especially those objects whose number is finite. Turning back, we see at least

four groups of objects that could be classified:

- the G-orbits of characteristics of nilpotent pairs;
- the G-orbits of wonderful nilpotent pairs;
- the G-orbits of even and almost even nilpotent pairs;
- the G-orbits of principal and almost principal nilpotent pairs.

The last group is the smallest one, and a complete description for it will be given in [ElPa]. But obtaining a classification of all characteristics requires at least a better understanding of possible fractional eigenspaces and, at the moment, I have no idea for this.

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